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Wigner trajectory characteristics in phase space and field theory

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Abstract. Exact characteristic trajectories are specified for the time-propagating Wigner phase-space distribution function. They are especially simple—indeed, classical—for the quantized simple harmonic oscillator, which serves as the underpinning of the field theoretic Wigner functional formulation introduced. Scalar field theory is thus reformulated in terms of distributions in field phase space. Applications to duality transformations in field theory are discussed.

An autonomous formulation of quantum mechanics, different from conventional Hilbert space or path integral quantization, is based on Wigner's phase-space distribution function (WF), which is a special representation of the density matrix [1]. In this formulation, known as deformation quantization [2], phase-space c-number functions are multiplied through the crucial noncommutative $\star$-product [3]. The empowering principle underlying this quantization is its operational isomorphism [2] to the conventional Heisenberg operator algebra of quantum mechanics.

Here, we employ the $\star$-unitary evolution operator, a '$\star$-exponential', to specify the time propagation of Wigner phase-space distribution functions. The answer is known to be remarkably simple for the harmonic oscillator WF, and consists of classical rotation in phase space for the full quantum system. It thus serves as the underpinning of the generalization to field theory we consider, in which the dynamics is specified through the evolution of c-number distributions on field phase space.

Wigner functions are defined by

$$f(x,p) = \frac{1}{2\pi}\int \mathrm{d}y\, \psi^*\left(x - \frac{\hbar}{2}y\right) \mathrm{e}^{-\mathrm{i}yp}\psi\left(x + \frac{\hbar}{2}y\right). \qquad (1)$$

Even though they amount to spatial auto-correlation functions of Schrödinger wavefunctions ψ, they can be determined without reference to such wavefunctions, in a logically autonomous structure. For instance, when the wavefunction is an energy (E) eigenfunction, the corresponding WF is time-independent and satisfies the two-sided energy $\star$-genvalue equations [4, 5],

$$H \star f = f \star H = Ef \qquad (2)$$

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where $\star$ is the essentially unique associative deformation of ordinary products on phase space,

$$\star \equiv \mathrm{e}^{\frac{\mathrm{i}\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)} \tag{3}$$

as defined by Groenewold [3], and developed in [2]. In practice, it may be evaluated through translations of function arguments, $f(x,p) \star g(x,p) = f(x + \frac{\mathrm{i}\hbar}{2}\overrightarrow{\partial}_p, p - \frac{\mathrm{i}\hbar}{2}\overrightarrow{\partial}_x) g(x,p)$, to produce pseudodifferential equations.

These WFs are real. They are bounded by the Schwarz inequality [8] to $-2/h \leqslant f \leqslant 2/h$. They can go negative, and, indeed, they do for all but Gaussian configurations, so they are not probability distributions [1]. However, upon integration over x or p, they yield marginal probability densities in p- and x-space, respectively. They can also be shown to be orthonormal [4, 5]. Unlike in Hilbert space quantum mechanics, naive superposition of solutions of the above does not hold, because of Baker's [8] fundamental nonlinear projection condition $f \star f = f/h$.

Time-dependence for WFs was succinctly specified by Moyal through the commutator bracket [6] bearing his name,

$$\mathrm{i}\hbar \frac{\partial}{\partial t} f(x,p;t) = H \star f(x,p;t) - f(x,p;t) \star H. \tag{4}$$

This turns out to be the essentially unique associative generalization of the Poisson bracket [7], to which it reduces as $\hbar \to 0$, yielding Liouville's theorem of classical mechanics, $\partial_t f + \{f, H\} = 0$.

For the evolution of the fundamental phase-space variables x and p, time evolution is simply the convective part of Moyal's equation, so the apparent sign is reversed, while the Moyal bracket actually reduces to the Poisson bracket. That is, the $\hbar$-dependence drops out, and these variables, in fact, evolve simply by the *classical* Hamilton equations of motion, $\dot{x} = \partial_p H$, $\dot{p} = -\partial_x H$.

What is the time-evolution of a WF like? This is the first question we address. Relying on the isomorphism to operator algebras of [2] indicated, one may solve for the time-trajectories of the WF, which turn out to be notably simple. By virtue of the $\star$-unitary evolution operator, a '$\star$-exponential' [2],

$$U_\star(x,p;t) = \mathrm{e}_\star^{\mathrm{i}tH/\hbar} \equiv 1 + (\mathrm{i}t/\hbar)H(x,p) + \frac{(\mathrm{i}t/\hbar)^2}{2!} H \star H + \frac{(\mathrm{i}t/\hbar)^3}{3!} H \star H \star H + \cdots \tag{5}$$

the time-evolved Wigner function is obtainable formally in terms of the Wigner function at $t = 0$ through associative combinatoric operations completely analogous to the conventional formulation of quantum mechanics of operators in Hilbert space. Specifically,

$$f(x,p;t) = U_\star^{-1}(x,p;t) \star f(x,p;0) \star U_\star(x,p;t). \tag{6}$$

As mentioned, the dynamical variables evolve classically,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x \star H - H \star x}{\mathrm{i}\hbar} = \partial_p H \tag{7}$$

and

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{p \star H - H \star p}{\mathrm{i}\hbar} = -\partial_x H. \tag{8}$$

Consequently, by associativity, the quantum evolution,

$$x(t) = U_\star \star x \star U_\star^{-1} \tag{9}$$

$$p(t) = U_\star \star p \star U_\star^{-1} \tag{10}$$

turns out to flow along *classical* trajectories.

What can one say about this formal time-evolution expression? Any WF in phase space, upon Fourier transformation resolves to

$$f(x,p) = \int \mathrm{d}a\, \mathrm{d}b\, \tilde{f}(a,b) \mathrm{e}^{\mathrm{i}ax} \mathrm{e}^{\mathrm{i}bp}. \tag{11}$$

However, note that exponentials of individual functions of x and p are also $\star$-exponentials of the same functions, or $\star$-versions of these functions, since the $\star$-product trivializes in the absence of a conjugate variable, so that

$$\mathrm{e}^{\mathrm{i}ax} \mathrm{e}^{\mathrm{i}bp} = \mathrm{e}_\star^{\mathrm{i}ax} \mathrm{e}_\star^{\mathrm{i}bp}. \tag{12}$$

Moreover, this is proportional to a $\star$-product, since

$$\mathrm{e}_\star^{\mathrm{i}ax} \star \mathrm{e}_\star^{\mathrm{i}bp} = \mathrm{e}_\star^{\mathrm{i}a(x+\mathrm{i}\hbar \vec{\partial}_p/2)} \mathrm{e}_\star^{\mathrm{i}bp} = \mathrm{e}_\star^{\mathrm{i}ax} \mathrm{e}_\star^{\mathrm{i}bp} \mathrm{e}^{-\mathrm{i}\hbar ab/2}. \tag{13}$$

Consequently, any Wigner function can be written as

$$f(x,p) = \int \mathrm{d}a\, \mathrm{d}b\, \tilde{f}(a,b) \mathrm{e}^{\mathrm{i}\hbar ab/2} \mathrm{e}_\star^{\mathrm{i}ax} \star \mathrm{e}_\star^{\mathrm{i}bp}. \tag{14}$$

It follows then, that, by insertion of $U_\star \star U_\star^{-1}$ pairs at every $\star$-multiplication, in general,

$$\begin{aligned} f(x,p;t) &= \int \mathrm{d}a\, \mathrm{d}b\, \tilde{f}(a,b) \mathrm{e}^{\mathrm{i}\hbar ab/2} \mathrm{e}_\star^{\mathrm{i}aU_\star^{-1}\star x \star U_\star} \star \mathrm{e}_\star^{\mathrm{i}bU_\star^{-1}\star p \star U_\star} \\ &= \int \mathrm{d}a\, \mathrm{d}b\, \tilde{f}(a,b) \mathrm{e}^{\mathrm{i}\hbar ab/2} \mathrm{e}_\star^{\mathrm{i}ax(-t)} \star \mathrm{e}_\star^{\mathrm{i}bp(-t)}. \end{aligned} \tag{15}$$

Unfortunately, in general, the above steps cannot be simply reversed to yield an integrand of the form $\tilde{f}(a,b)\mathrm{e}^{\mathrm{i}ax(-t)}\mathrm{e}^{\mathrm{i}bp(-t)}$. But, in some fortuitous circumstances, they can, and in this case the evolution of the Wigner function reduces to merely backward evolution of its arguments x, p along classical trajectories, while its functional form itself remains unchanged:

$$f(x,p;t) = f(x(-t), p(-t); 0). \tag{16}$$

To illustrate this, consider the simple linear harmonic oscillator (taking $m = 1$, $\omega = 1$),

$$H = \frac{p^2 + x^2}{2} = \frac{x - \mathrm{i}p}{\sqrt{2}} \star \frac{x + \mathrm{i}p}{\sqrt{2}} + \frac{\hbar}{2}. \tag{17}$$

It is easily seen that

$$\mathrm{i}\hbar\dot{x} = x \star H - H \star x = \mathrm{i}\hbar p \qquad \mathrm{i}\hbar\dot{p} = p \star H - H \star p = -\mathrm{i}\hbar x \tag{18}$$

and thus the canonical variables indeed evolve classically:

$$\begin{aligned} X &\equiv x(t) = U_\star \star x \star U_\star^{-1} = x\cos t + p \sin t \\ P &\equiv p(t) = U_\star \star p \star U_\star^{-1} = p \cos t - x \sin t. \end{aligned} \tag{19}$$

This checks against the $\star$-exponential for the SHO, [2], $\mathrm{e}_\star^{\mathrm{i}tH/\hbar} = \frac{1}{\cos(t/2)} \exp(2\mathrm{i}\tan(t/2)H/\hbar)$.

Now, recall the degenerate case of the Baker–Campbell–Hausdorff combinatoric identity for any two operators with *constant* commutator with respect to any associative multiplication, thus for any phase-space functions ξ and η under $\star$-multiplication. If

$$\xi \star \eta - \eta \star \xi = c \tag{20}$$

then,

$$\mathrm{e}_\star^{\xi} \star \mathrm{e}_\star^{\eta} = \mathrm{e}_\star^{\xi+\eta} \mathrm{e}^{c/2}. \tag{21}$$

Application of this identity as well as (13) and (12) directly yields

$$\begin{aligned} \mathrm{e}_\star^{\mathrm{i}ax(-t)} \star \mathrm{e}_\star^{\mathrm{i}bp(-t)} \mathrm{e}^{\mathrm{i}\hbar ab/2} &= \mathrm{e}_\star^{\mathrm{i}(a\cos t+b\sin t)x+\mathrm{i}(b\cos t-a\sin t)p} \\ &= \mathrm{e}_\star^{\mathrm{i}(a\cos t+b\sin t)x} \star \mathrm{e}_\star^{\mathrm{i}(b\cos t-a\sin t)p} \mathrm{e}^{\mathrm{i}\hbar(a\cos t+b\sin t)(b\cos t-a\sin t)/2} \\ &= \mathrm{e}_\star^{\mathrm{i}(a\cos t+b\sin t)x} \mathrm{e}_\star^{\mathrm{i}(b\cos t-a\sin t)p} \\ &= \mathrm{e}^{\mathrm{i}(a\cos t+b\sin t)x} \mathrm{e}^{\mathrm{i}(b\cos t-a\sin t)p}. \end{aligned} \tag{22}$$

Consequently,

$$f(x,p;t) = \int \mathrm{d}a\,\mathrm{d}b\,\tilde{f}(a,b)\mathrm{e}^{\mathrm{i}ax(-t)}\mathrm{e}^{\mathrm{i}bp(-t)} \tag{23}$$

and hence the reverse convective flow (16) obtains.

The result for the SHO is the preservation of the functional form of the Wigner distribution function along classical phase-space trajectories:

$$f(x,p;t) = f(x\cos t - p\sin t,\, p\cos t + x\sin t; 0). \tag{24}$$

What this means is that *any* Wigner distribution rotates uniformly on the phase plane around the origin, essentially classically, even though it provides a complete quantum mechanical description. Note how, in general, this result is deprived of importance, or, at the very least, simplicity, upon integration in x (or p) to yield probability densities: the rotation induces shape variations of the oscillating probability density profile. Only if, as is the case for coherent states [10], a Wigner function configuration has an additional axial x–p symmetry around its *own* centre, will it possess an invariant profile upon this rotation, and hence a shape-invariant oscillating probability density.

The result (24), of course, is not new. It was clearly recognized by Wigner [11]. It follows directly from (4) for (17) that

$$(\partial_t + p\partial_x - x\partial_p)f(x,p;t) = 0. \tag{25}$$

The characteristics of this partial differential equation correspond to the above uniform rotation in phase space, so it is easily seen to be solved by (24). The result was given explicitly in [3] and also [9], following different derivations. Lesche [12], has also reached this result in an elegant fifth derivation, by noting that for quadratic Hamiltonians such as this one, the linear rotation of the dynamical variables (19) leaves the symplectic quadratic form invariant, and thus the $\star$-product invariant. That is, the gradients in the $\star$-product may also be taken to be with respect to the time-evolved canonical variables (19), X and P; hence, after inserting $U_\star \star U_\star^{-1}$ in the $\star$-functional form of f, the $\star$-products may be taken to be with respect to X and P, and the functional form of f is preserved, (16). This only holds for quadratic Hamiltonians.

Dirac's interaction representation may then be based on this property, for a general Hamiltonian consisting of a basic SHO part, $H_0 = (p^2+x^2)/2$, supplemented by an interaction part,

$$H = H_0 + H_I. \tag{26}$$

Now, upon defining

$$w \equiv \mathrm{e}_\star^{\mathrm{i}tH_0/\hbar} \star f \star \mathrm{e}_\star^{-\mathrm{i}tH_0/\hbar} \tag{27}$$

it follows that Moyal's evolution equation reads,

$$\mathrm{i}\hbar\frac{\partial}{\partial t}w(x,p;t) = \mathcal{H}_I \star w(x,p;t) - w(x,p;t) \star \mathcal{H}_I \tag{28}$$

where$\mathcal{H}_I \equiv \mathrm{e}_\star^{\mathrm{i}tH_0/\hbar} \star H_I \star \mathrm{e}_\star^{-\mathrm{i}tH_0/\hbar}$. Expressing H_I as a $\star$-function leads to a simplification.

In terms of the convective variables (19), X, P, $\mathcal{H}_I(x, p) = H_I(X, P)$, and $w(x, p; t) = f(X, P; t)$, while $\star$ may refer to these convective variables as well. Finally, then,

$$\mathrm{i}\hbar \frac{\partial}{\partial t} f(X, P; t) = H_I(X, P) \star f(X, P; t) - f(X, P; t) \star H_I(X, P). \tag{29}$$

In the uniformly rotating frame of the convective variables, the WF time-evolves according to the interaction Hamiltonian—while, for vanishing interaction Hamiltonian, $f(X, P; t)$ is constant and yields (24). Below, in generalizing to field theory, this provides the basis of the interaction picture of perturbation theory, where the basis canonical fields evolve classically as above†.

To produce Wigner functionals in scalar field theory, one may start from the standard, noncovariant, formulation of field theory in Hilbert space, in terms of Schrödinger wavefunctionals.

For a free-field Hamiltonian, the energy eigenfunctionals are Gaussian in form. For instance, without loss of generality, in two dimensions (x is a spatial coordinate, and $t = 0$ in all fields), the ground state functional is

$$\Psi[\phi] = \exp\left(-\frac{1}{2\hbar}\int \mathrm{d}x\, \phi(x)\sqrt{m^2 - \nabla_x^2}\, \phi(x)\right). \tag{30}$$

Boundary conditions are assumed such that the $\sqrt{m^2 - \nabla_x^2}$ kernel in the exponent is naively self-adjoint. 'Integrating by parts' one of the $\sqrt{m^2 - \nabla_z^2}$ kernels, functional derivation $\delta\phi(x)/\delta\phi(z) = \delta(z - x)$ then leads to

$$\hbar \frac{\delta}{\delta\phi(z)}\Psi[\phi] = -\left(\sqrt{m^2 - \nabla_z^2}\, \phi(z)\right)\Psi[\phi] \tag{31}$$

$$\hbar^2 \frac{\delta^2}{\delta\phi(w)\delta\phi(z)}\Psi[\phi] = \left(\sqrt{m^2 - \nabla_w^2}\phi(w)\right) \times \left(\sqrt{m^2 - \nabla_z^2}\phi(z)\right)\Psi[\phi] - \hbar\sqrt{m^2 - \nabla_z^2}\delta(w - z)\Psi[\phi]. \tag{32}$$

Note that the divergent zero-point energy density,

$$E_0 = \frac{\hbar}{2}\lim_{w\to z}\sqrt{m^2 - \nabla_z^2}\, \delta(w - z) \tag{33}$$

may be handled rigorously using ζ-function regularization.

Leaving this zero-point energy present, leads to the standard energy eigenvalue equation, again through integration by parts,

$$\frac{1}{2}\int \mathrm{d}z\, \left(-\hbar^2 \frac{\delta^2}{\delta\phi(z)^2} + \phi(z)(m^2 - \nabla_z^2)\phi(z)\right)\Psi[\phi] = E_0\, \Psi[\phi]. \tag{34}$$

A natural adaptation to the corresponding Wigner functional is the following. For a functional measure $[\mathrm{d}\eta/2\pi] = \prod_x \mathrm{d}\eta(x)/2\pi$, one obtains

$$W[\phi, \pi] = \int \left[\frac{\mathrm{d}\eta}{2\pi}\right]\Psi^*\left[\phi - \frac{\hbar}{2}\eta\right]\mathrm{e}^{-\mathrm{i}\int \mathrm{d}x\, \eta(x)\pi(x)}\Psi\left[\phi + \frac{\hbar}{2}\eta\right] \tag{35}$$

where $\pi(x)$ is to be understood as the local field variable canonically conjugate to $\phi(x)$. However, in this expression, both ϕ and π are *classical* variables, not operator-valued fields, in full analogy to the phase-space quantum mechanics already discussed.

† [13] discusses field theoretic interaction representations in phase space, which do not appear coincident with the present one.

For the Gaussian ground state wavefunctional, this evaluates to

$$
\begin{aligned}
W[\phi,\pi] = \int \left[\frac{d\eta}{2\pi}\right] \exp\left(-\frac{1}{2\hbar}\int dx \left(\phi(x)-\frac{\hbar}{2}\eta(x)\right)\sqrt{m^2-\nabla_x^2}\left(\phi(x)-\frac{\hbar}{2}\eta(x)\right)\right) \\
\times e^{-i\int dx\,\eta(x)\pi(x)} \exp\left(-\frac{1}{2\hbar}\int dx \left(\phi(x)+\frac{\hbar}{2}\eta(x)\right)\right. \\
\left.\times \sqrt{m^2-\nabla_x^2}\left(\phi(x)+\frac{\hbar}{2}\eta(x)\right)\right) \\
= \exp\left(-\frac{1}{\hbar}\int dx\,\phi(x)\sqrt{m^2-\nabla_x^2}\phi(x)\right) \\
\times\left(\int\left[\frac{d\eta}{2\pi}\right] e^{-i\int dx\,\eta(x)\pi(x)} \exp\left(-\frac{\hbar}{4}\int dx\,\eta(x)\sqrt{m^2-\nabla_x^2}\eta(x)\right)\right).
\end{aligned}
\tag{36}
$$

So

$$
W[\phi,\pi] = \mathcal{N}\exp\left(-\frac{1}{\hbar}\int dx\left(\left(\phi(x)\sqrt{m^2-\nabla_x^2}\phi(x)\right)+\left(\pi(x)\left(\sqrt{m^2-\nabla_x^2}\right)^{-1}\pi(x)\right)\right)\right)
\tag{37}
$$

where $\mathcal{N}$ is a normalization factor. It is the expected collection of harmonic oscillators.

This Wigner functional is, of course [5], an energy $\star$-genfunctional, also checked directly. For

$$
H_0[\phi,\pi] \equiv \tfrac{1}{2}\int dx\,(\pi(x)^2+\phi(x)(m^2-\nabla_x^2)\phi(x))
\tag{38}
$$

and the inevitable generalization

$$
\star \equiv \exp\left(\frac{i\hbar}{2}\int dx\left(\frac{\overleftarrow{\delta}}{\delta\phi(x)}\frac{\overrightarrow{\delta}}{\delta\pi(x)} - \frac{\overleftarrow{\delta}}{\delta\pi(x)}\frac{\overrightarrow{\delta}}{\delta\phi(x)}\right)\right)
\tag{39}
$$

it follows that

$$
\begin{aligned}
H_0 \star W = \int \frac{dx}{2}\left(\left(\pi(x)-\frac{1}{2}i\hbar\frac{\delta}{\delta\phi(x)}\right)^2\right. \\
\left.+\left(\phi(x)+\frac{1}{2}i\hbar\frac{\delta}{\delta\pi(x)}\right)(m^2-\nabla_x^2)\left(\phi(x)+\frac{1}{2}i\hbar\frac{\delta}{\delta\pi(x)}\right)\right)W[\phi,\pi] \\
= \int\frac{dx}{2}\left(\pi(x)^2-\frac{1}{4}\hbar^2\frac{\delta}{\delta\pi(x)}(m^2-\nabla_x^2)\frac{\delta}{\delta\pi(x)}\right. \\
\left.+\phi(x)(m^2-\nabla_x^2)\phi(x)-\frac{1}{4}\hbar^2\frac{\delta^2}{\delta\phi(x)^2}\right)W[\phi,\pi] \\
= E_0 W[\phi,\pi].
\end{aligned}
\tag{40}
$$

This is indeed the ground state Wigner energy-$\star$-genfunctional. The $\star$-genvalue is again the zero-point energy, which could have been removed by point-splitting the energy density, as indicated earlier. There does not seem to be a simple point-splitting procedure that regularizes the $\star$-product as defined above and also preserves associativity.

As in the case of the SHO discussed above, free-field time-evolution for Wigner functionals is also effected by Dirac delta functionals whose support lies on the classical field time evolution equations. Fields evolve according to the equations,

$$
-i\hbar\partial_t\phi = H\star\phi-\phi\star H \qquad -i\hbar\partial_t\pi = H\star\pi-\pi\star H.
\tag{41}
$$

For H_0, these equations are the classical evolution equations for free fields,

$$\partial_t \phi(x,t) = \pi(x,t) \qquad \partial_t \pi(x,t) = -(m^2 - \nabla_x^2)\phi(x,t). \tag{42}$$

Formally, the solutions are represented as

$$\phi(x,t) = \cos\left(t\sqrt{m^2-\nabla_x^2}\right)\phi(x,0) + \sin\left(t\sqrt{m^2-\nabla_x^2}\right)\frac{1}{\sqrt{m^2-\nabla_x^2}}\pi(x,0) \tag{43}$$

$$\pi(x,t) = -\sin\left(t\sqrt{m^2-\nabla_x^2}\right)\sqrt{m^2-\nabla_x^2}\phi(x,0) + \cos\left(t\sqrt{m^2-\nabla_x^2}\right)\pi(x,0). \tag{44}$$

From these, it follows by the functional chain rule that

$$\begin{aligned}\int \mathrm{d}x \left(\pi(x,0)\frac{\delta}{\delta\phi(x,0)} - ((m^2-\nabla_x^2)\phi(x,0))\frac{\delta}{\delta\pi(x,0)}\right) \\ = \int \mathrm{d}x \left(\pi(x,t)\frac{\delta}{\delta\phi(x,t)} - ((m^2-\nabla_x^2)\phi(x,t))\frac{\delta}{\delta\pi(x,t)}\right)\end{aligned} \tag{45}$$

for any time t.

Consider the free-field Moyal evolution equation for a generic (not necessarily energy-$\star$-genfunctional) WF, corresponding to (25),

$$\partial_t W = -\int \mathrm{d}x \left(\pi(x)\frac{\delta}{\delta\phi(x)} - \phi(x)(m^2-\nabla_x^2)\frac{\delta}{\delta\pi(x)}\right) W. \tag{46}$$

The solution is

$$W[\phi,\pi;t] = W[\phi(-t),\pi(-t);0]. \tag{47}$$

Adapting the method of characteristics for first-order equations to a functional context, one may simply check this solution again using the chain rule for functional derivatives, and the field equations *evolved backwards in time* as specified:

$$\begin{aligned}\partial_t W[\phi,\pi;t] &= \partial_t W[\phi(-t),\pi(-t);0] \\ &= \int \mathrm{d}x \left(\partial_t\phi(x,-t)\frac{\delta}{\delta\phi(x,-t)} + \partial_t\pi(x,-t)\frac{\delta}{\delta\pi(x,-t)}\right) W[\phi(-t),\pi(-t);0] \\ &= \int \mathrm{d}x \left((-\pi(x,-t))\frac{\delta}{\delta\phi(x,-t)} + ((m^2-\nabla_x^2)\phi(x,-t))\frac{\delta}{\delta\pi(x,-t)}\right) \\ &\quad \times W[\phi(-t),\pi(-t);0] \\ &= \int \mathrm{d}x \left((-\pi(x,-t))\frac{\delta}{\delta\phi(x,-t)} + ((m^2-\nabla_x^2)\phi(x,-t))\frac{\delta}{\delta\pi(x,-t)}\right) \\ &\quad \times W[\phi,\pi;t] \\ &= -\int \mathrm{d}x \left(\pi(x)\frac{\delta}{\delta\phi(x)} - (m^2-\nabla_x^2)\phi(x)\frac{\delta}{\delta\pi(x)}\right) W[\phi,\pi;t].\end{aligned} \tag{48}$$

The quantum Wigner functional for free fields time-evolves along classical field configurations. In complete analogy to the interaction representation for single particle quantum mechanics, (29), the perturbative series in the interaction Hamiltonian (written as a $\star$-function of fields) is then defined in terms of convective (time-evolved free field) variables Φ, Π:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}W[\Phi,\Pi;t] = H_I[\Phi,\Pi]\star W[\Phi,\Pi;t] - W[\Phi,\Pi;t]\star H_I[\Phi,\Pi]. \tag{49}$$

In [5], a transformation function T was introduced to accommodate arbitrary canonical transformations induced by a generating function F in quantum mechanics, following Dirac. The WF in terms of the canonically transformed variables is obtained by convolving with this transformation function. In complete analogy, in scalar field theory, cf [15], given a canonical transformation from field variables ϕ, π to variables φ, ϖ effected by a generating functional $F[\phi, \varphi]$, one may deduce that the WF in terms of the canonically transformed field variables is

$$W[\phi, \pi] = \int \left[\frac{\mathrm{d}\varphi\, \mathrm{d}\varpi}{2\pi}\right] T[\phi, \pi; \varphi, \varpi] \mathcal{W}[\varphi, \varpi] \tag{50}$$

where

$$T[\phi, \pi; \varphi, \varpi] = \int \left[\frac{\mathrm{d}\eta\, \mathrm{d}\rho}{2\pi}\right] \exp \mathrm{i}\Bigg(F\left[\phi + \frac{1}{2}\eta, \varphi + \frac{1}{2}\rho\right] - \mathrm{i}F^* \left[\phi - \frac{1}{2}\eta, \varphi - \frac{1}{2}\rho\right] + \int \mathrm{d}x\, (\varpi(x)\rho(x) - \pi(x)\eta(x))\Bigg). \tag{51}$$

For example, the generating functional for free field duality between a two-dimensional space-time scalar φ and a pseudoscalar ϕ is

$$F[\phi, \varphi] = \int \mathrm{d}x\, \phi \partial_x \varphi \tag{52}$$

so it yields the classical canonical transformations

$$\pi = \frac{\delta}{\delta\phi} F = \partial_x \varphi \qquad \varpi = -\frac{\delta}{\delta\varphi} F = \partial_x \phi. \tag{53}$$

After some computation, it follows that

$$T[\phi, \pi; \varphi, \varpi] = [2\pi]\delta[\partial_x \varphi - \pi]\delta[\varpi - \partial_x \phi]. \tag{54}$$

The ensuing relation between the respective dual Wigner functionals is then quite simple:

$$W[\phi, \pi] = \mathcal{W}\left[\int^x \pi, \partial_x \phi\right]. \tag{55}$$

A less exceptional example is the canonical transformation from the chiral σ-model in two dimensions to its dual counterpart, [14, 15] generated by

$$F = \int \mathrm{d}x\, \phi^i J^i(\varphi) \tag{56}$$

where

$$J^i[\varphi] = \sqrt{1 - \varphi^2}\, \overleftrightarrow{\partial_x}\, \varphi^i + \varepsilon^{ijk} \varphi^j \partial_x \varphi^k \tag{57}$$

is the spatial component of the right $(V + A)$ current. The resulting classical relations are of course more complicated than for the free field duality above [15]. For example, the nonlinear relation between field-conjugate and dual field is $\pi^i = \frac{\partial}{\partial\phi} F = J^i[\varphi]$. Even so, the resulting quantum transformation functional is considerably more complicated beyond this nonlinearity. Only one functional integration (over η) is trivially carried out in the previous general expression for T in terms of F, leading to

$$T = \int [\mathrm{d}\rho] \delta[\pi^i - \tfrac{1}{2} J^i(\varphi + \tfrac{1}{2}\rho) - \tfrac{1}{2} J^i(\varphi - \tfrac{1}{2}\rho)] \times \exp\left(\mathrm{i} \int \mathrm{d}x\, \phi^i (J^i(\varphi + \tfrac{1}{2}\rho) - J^i(\varphi - \tfrac{1}{2}\rho))\right) \exp\left(\mathrm{i} \int \mathrm{d}x\, \varpi\rho\right). \tag{58}$$

The delta-functional appearing explicitly here does not enforce the aforementioned classical constraint, but rather a ‘quantum arithmetically-averaged’ form of it. The other classical constraint, involving ϖ, is obliterated by the remaining functional integral over ρ, and emerges clearly only in the weak-φ-field limit, where $J^i(\varphi + \frac{1}{2}\rho) - J^i(\varphi - \frac{1}{2}\rho) \simeq \partial_x \rho^i$. This is more typical of quantum effects in field theory.

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